Lecture 5

Monday, 7 September 2026 8:22 PM

Today: **Sperner's Lemma**

But first some convex geometry:

- **Affine hull**: given vectors $v_1, \dots, v_n \in \mathbb{R}^d$, the affine hull
$$= \left\{ x = \sum_{i=1}^{k} \lambda_i v_i \,,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

- **Affine independence**: vectors $v_1, \dots, v_n \in \mathbb{R}^d$ are affinely independent if no vector is in the affine hull of the others.
Equivalently: $v_1, \dots, v_n$ are affinely independent if
$$\sum_{i=1}^{n} \lambda_i v_i = 0 \,,\ \sum_{i=1}^{n} \lambda_i = 0 \Rightarrow \forall i \ \lambda_i = 0$$

- A **simplex** is the convex hull of n affinely independent vectors
i.e. $$\sigma = \left\{ \sum_{i=1}^{n} \lambda_i v_i ;\ \sum_{i=1}^{n} \lambda_i = 1,\ \forall i \ \lambda_i \geq 0 \right\}$$
$v_1, \dots, v_n$ are affinely independent
(note that σ has dimension $n-1$)

Claim: $\forall x \in \sigma$, $\exists$ unique $\lambda_1, \dots, \lambda_n$ s.t. $x = \sum_{i=1}^{n} \lambda_i v_i$, $\sum_i \lambda_i = 1$, $\forall i, \lambda_i \geq 0$.

- Δ_{n-1} is the **standard simplex** $= \left\{ \sum_{i=1}^{n} \lambda_i e_i \,,\ \sum_{i=1}^{n} \lambda_i = 1,\ \forall i \ \lambda_i \geq 0 \right\}$
$e_i = (0, 0 \dots, 1, 0, \dots, 0)$
$\uparrow$ ith posn.

Note: $\Delta_{n-1} \in \mathbb{R}^n$, has n vertices, but has dimension $n-1$.

For $x \in \Delta_{n-1}$, $\mathrm{Supp}(x) = \{ i : x_i > 0 \}$

More generally, for $x \in \sigma$, $\mathrm{Supp}(x) = \{ i : \lambda_i > 0 \}$

Face: CH of a subset of vertices $S \subseteq [n]$
For a simplex σ, subset S of vertices $v_1, \dots, v_n$,
σ_S : $CH \{ v_i : i \in S \}$

Facet: A facet is a face of dimension $n-2$, equivalently given by the CH of $n-1$ vertices. So there are exactly n facets in a simplex.
For a facet, we call the missing vertex as the vertex opposite the facet.

relative interior $\mathrm{relint}(\sigma) = \{ x : \mathrm{supp}(x) = [n] \}$
boundary $\partial(\sigma) = \sigma \setminus \mathrm{relint}(\sigma) = \{ x : \mathrm{supp}(x) \neq [n] \}$

Triangulation (of σ): A collection of simplices of dimension $(n-1)$, so that:
① their union is σ,
② any two simplices in the collection either have no intersection, or intersect in a common face of both.
These are called **elementary simplices**.

Claim: Let τ be a facet of an elementary simplex σ of Δ_{n-1}. Then:
① if $\tau \subseteq \partial(\Delta_{n-1})$ then τ is a facet of exactly one elementary simplex
② else τ is a facet of exactly 2 elementary simplices.
③ No two facets lie in $\partial(\Delta_{n-1})$, or are shared w/ the same simplex.

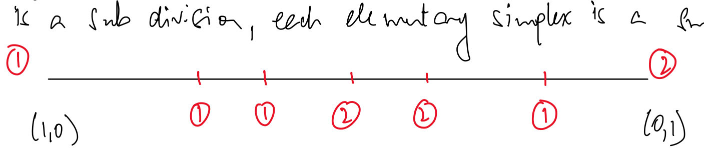
Sperner Labelling:

Consider the simplex Δ_{n-1}, triangulation T (with elementary simplices σ).
Let $V(T)$ be vertices of all the simplices.
Then $l : V(T) \to [n]$ is a Sperner labelling if
$l(x) \in \{ i : x_i > 0 \}$

An elementary simplex is called a rainbow simplex if its vertices have all n colours.

Sperner's Lemma: There are an odd number of rainbow simplices (thus, at least 1).

Proof: By induction. Take $n=2$. Then a simplex is a line segment, a triangulation is a subdivision, each elementary simplex is a smaller segment



Can prove: # rainbow simplices = # segments w/ both labels = odd.

For $n > 2$. Fix a coordinate, say n.
Consider a graph with ① a vertex for every elementary simplex,
② one vertex for the standard simplex (say S)
Edges: for 2 elementary simplices, $\exists$ an edge b/w them if they have a common facet (simplex of dimension $n-2$) labelled $\{1, 2, \dots, n-1\}$
edge b/w standard simplex & an elementary simplex if the elementary simplex has a facet $\tau \subseteq \partial(\Delta_{n-1})$

Claim: In the graph described:
① S has odd degree
② every rainbow simplex has degree 1.
③ every simplex labelled $\{1, \dots, n-1\}$ has degree 2
④ every other simplex has degree 0.

The proof of Sperner's lemma follows immediately.

Proof of claim: Call a facet labelled $1 \dots n-1$ a "door"
② is immediate: there is a single door, and by earlier claim, this facet τ is either $\subseteq \partial(\Delta_{n-1})$ (in which case this simplex has an edge to S), or is common to another simplex, in which case these two simplices have an edge
③ Let σ be a simplex w/ labels $1 \dots n-1$. Since there are n vertices, some label (say 1) appears exactly twice. So there are exactly two doors, giving degree 2
④ Any other simplex does not have labels $1, \dots, n-1$, so no doors.
① Let (S, σ) be an edge. Then σ has a facet τ labelled $1 \dots n-1$, & this $\tau \subseteq \partial(\Delta_{n-1})$. Further this must lie on the facet $\hat{\tau}$ of Δ_{n-1} spanned by $e_1, \dots, e_{n-1}$.
This facet is a simplex of dimension $n-2$, hence has an odd # of elementary simplices that are fully-labelled.
Each such elementary simplex is a door, hence S has odd degree ∎
